

On the Cartesian Skeleton and the Factorization of the Strong Product of Digraphs

Marc Hellmuth^{a,*}, Tilen Marc^{a,b}

^aCenter for Bioinformatics, Saarland University, Building E 2.1, Room 413, P.O. Box 15 11 50, D-66041 Saarbrücken, Germany

^bFaculty of Mathematics and Physics, University of Ljubljana Jadranska 19, 1000 Ljubljana, Slovenia

Abstract

The three standard products (the Cartesian, the direct and the strong product) of undirected graphs have been well-investigated, unique prime factor decomposition (PFD) are known and polynomial time algorithms have been established for determining the prime factors.

For directed graphs, unique PFD results with respect to the standard products are known. However, there is still a lack of algorithms, that computes the PFD of directed graphs with respect to the direct and the strong product in general. In this contribution, we focus on the algorithmic aspects for determining the PFD of directed graphs with respect to the strong product. Essential for computing the prime factors is the construction of a so-called Cartesian skeleton. This article introduces the notion of the Cartesian skeleton of directed graphs as a generalization of the Cartesian skeleton of undirected graphs. We provide new, fast and transparent algorithms for its construction. Moreover, we present a first polynomial time algorithm for determining the PFD with respect to the strong product of arbitrary connected digraphs.

Keywords: Directed Graph, Strong Product, Prime Factor Decomposition Algorithms, Dispensable, Cartesian Skeleton

1. Introduction

Graphs and in particular graph products arise in a variety of different contexts, from computer science [1, 20] to theoretical biology [26, 28], computational engineering [21, 22, 23] or just as natural structures in discrete mathematics [7, 16].

For undirected simple graphs, it is well-known that each of the three standard graph products, the Cartesian product [4, 19, 25, 27], the direct product [15, 24] and the strong product [2, 5, 24], satisfies the unique prime factor decomposition property under certain conditions, and there are polynomial-time algorithms to determine the prime factors. Several monographs cover the topic in substantial detail and serve as standard references [7, 16].

For directed graphs, or digraphs for short, only partial results are known. Feigenbaum showed that the Cartesian product of digraphs satisfies the unique prime factorization property and provided a polynomial-time algorithm for its computation [3]. McKenzie proved that digraphs have a unique prime factor decomposition w.r.t. direct product requiring strong conditions on connectedness [24]. This result was extended by Imrich and Klöckl in [17, 18]. The authors provided unique prime factorization theorems and a polynomial-time algorithm for the direct product of digraphs under relaxed connectivity, but additional so-called thinness conditions. The results of McKenzie also imply that the strong product of digraphs can be uniquely decomposed into prime factors [24]. Surprisingly, so far no general algorithm for determining the prime factors of the strong product of digraphs has been established.

In this contribution, we are concerned with the algorithmic aspect of the *prime factor decomposition*, PFD for short, w.r.t. the strong product of digraphs. The key idea for the prime factorization of a strong product digraph

*corresponding author

Email addresses: marc.hellmuth@bioinf.uni-sb.de (Marc Hellmuth), marct15@gmail.com (Tilen Marc)

$G = H \boxtimes K$ is the same as for undirected graphs: We define the Cartesian skeleton $\mathbb{S}(G)$ of G . The Cartesian skeleton $\mathbb{S}(G)$ is decomposed with respect to the Cartesian product of digraphs. Afterwards, one determines the prime factors of G w.r.t. the strong product, using the information of the PFD of $\mathbb{S}(G)$. This approach can easily be extended if G is not S -thin. In this contribution, we introduce the notion of the Cartesian skeleton of directed graphs and show that it satisfies $\mathbb{S}(H \boxtimes K) = \mathbb{S}(H) \square \mathbb{S}(K)$ for so-called “ S -thin” digraphs. We prove that $\mathbb{S}(G)$ is connected whenever G is connected and provide new, fast and transparent algorithms for its construction. Furthermore, we present the first polynomial-time algorithm for the computation of the PFD w.r.t. the strong product of arbitrary connected digraphs.

2. Preliminaries

2.1. Basic Notation

A *digraph* $G = (V, E)$ is a tuple consisting of a set of vertices $V(G) = V$ and a set of ordered pairs $xy \in E(G) = E$, called (directed) edges or arcs. In the sequel we consider only simple digraphs with finite vertex and edge set. It is possible that both, xy and yx are contained in E . However, we only consider digraphs without loops, i.e., $xx \notin E$ for all $x \in V$. An *undirected graph* $G = (V, E)$ is a tuple consisting of a set of vertices $V(G) = V$ and a set of unordered pairs $\{x, y\} \in E(G) = E$. The *underlying undirected graph* of a digraph $G = (V, E)$ is the graph $U(G) = (V, F)$ with edge set $F = \{\{x, y\} \mid xy \in E \text{ or } yx \in E\}$. A digraph H is a *subgraph* of a digraph G , in symbols $H \subseteq G$, if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. If in addition $V(H) = V(G)$, we call H a *spanning subgraph* of G . If $H \subseteq G$ and all pairs of adjacent vertices in G are also adjacent in H then H is called an *induced subgraph*. The digraph $K_n = (V, E)$ with $|V| = n$ and $E = V \times V \setminus \{(x, x) \mid x \in V\}$ is called a *complete graph*.

A map $\gamma : V(H) \rightarrow V(G)$ such that $xy \in E(H)$ implies $\gamma(x)\gamma(y) \in E(G)$ for all $x, y \in V(H)$ is a *homomorphism*. We call two digraphs G and H *isomorphic*, and write $G \cong H$, if there exists a bijective homomorphism γ whose inverse function is also a homomorphism. Such a map γ is called an *isomorphism*.

Let $G = (V, E)$ be a digraph. The (closed) N^+ -neighborhood or *out-neighborhood* $N^+[v]$ of a vertex $v \in V$ is defined as $N^+[v] = \{x \mid vx \in E\} \cup \{v\}$. Analogously, the N^- -neighborhood or *in-neighborhood* $N^-[v]$ of a vertex $v \in V$ is defined as $N^-[v] = \{x \mid xv \in E\} \cup \{v\}$. If there is a risk of confusion we will write N_G^+ , resp., N_G^- to indicate that the respective neighborhoods are taken w.r.t. G . The *maximum degree* Δ of a digraph $G = (V, E)$ is defined by $\max_{v \in V} (|N^+[v] \setminus \{v\}| + |N^-[v] \setminus \{v\}|)$.

A digraph $G = (V, E)$ is *weakly connected*, or *connected* for short, if for every pair $x, y \in V$ there exists a sequence $w = (x_0, \dots, x_n)$, called *walk (connecting x and y)* or just *xy -walk*, with $x = x_0, y = x_n$ such that $x_i x_{i+1} \in E$ or $x_{i+1} x_i \in E$ for all $i \in \{0, \dots, n-1\}$. In other words, we call a digraph connected whenever its underlying undirected graph is connected.

2.2. The Cartesian and Strong Product

The vertex set of the *strong product* $G_1 \boxtimes G_2$ of two digraphs G_1 and G_2 is defined as $V(G_1) \times V(G_2) = \{(v_1, v_2) \mid v_1 \in V(G_1), v_2 \in V(G_2)\}$. Two vertices $(x_1, x_2), (y_1, y_2)$ are adjacent in $G_1 \boxtimes G_2$ if one of the following conditions is satisfied:

- (i) $x_1 y_1 \in E(G_1)$ and $x_2 = y_2$,
- (ii) $x_2 y_2 \in E(G_2)$ and $x_1 = y_1$,
- (iii) $x_1 y_1 \in E(G_1)$ and $x_2 y_2 \in E(G_2)$.

The *Cartesian product* $G_1 \square G_2$ has the same vertex set as $G_1 \boxtimes G_2$, but vertices are only adjacent if they satisfy (i) or (ii). Consequently, the edges of a strong product that satisfy (i) or (ii) are called *Cartesian*, the others *non-Cartesian*.

The one-vertex complete graph K_1 serves as a unit for both products, as $K_1 \square G = G$ and $K_1 \boxtimes G = G$ for all graphs G . It is well-known that both products are associative and commutative, see [7]. Hence, a vertex x of the strong product $\boxtimes_{i=1}^n G_i$ is properly “coordinatized” by the vector (x_1, \dots, x_n) whose entries are the vertices x_i of its factor graphs G_i . Therefore, the endpoints of a Cartesian edge in a strong product differ in exactly one coordinate.

The Cartesian product and the strong product of digraphs is connected if and only if each of its factors is connected [7].

In the product $\boxtimes_{i=1}^n G_i$, a G_j -layer through vertex x with coordinates (x_1, \dots, x_n) is the induced subgraph G_j^x in G with vertex set $\{(x_1, \dots, x_{j-1}, v, x_{j+1}, \dots, x_n) \in V(G) \mid v \in V(G_j)\}$. Thus, G_j^x is isomorphic to the factor G_j for every $x \in V(G)$. For $y \in V(G_j^x)$ we have $G_j^x = G_j^y$, while $V(G_j^x) \cap V(G_j^z) = \emptyset$ if $z \notin V(G_j^x)$.

Finally, it is well-known that both products of connected digraphs satisfy the unique prime factorization property.

Theorem 2.1 ([3]). *Every finite simple connected digraph has a unique representation as a Cartesian product of prime digraphs, up to isomorphism and order of the factors.*

Theorem 2.2 ([24]). *Every finite simple connected digraph has a unique representation as a strong product of prime digraphs, up to isomorphism and order of the factors.*

In the sequel of this paper we will make frequent use of the fact that for $G = G_1 \boxtimes G_2$ holds $N_G^+(x, y) = N_{G_1}^+[x] \times N_{G_2}^+[y]$ and $N_G^-(x, y) = N_{G_1}^-[x] \times N_{G_2}^-[y]$.

2.3. The Relations S^+ , S^- and S and Thinness

It is important to notice that although the PFD w.r.t. the strong product of connected digraphs is unique, the assignment of an edge being Cartesian or non-Cartesian is not unique, in general. This is usually possible if two vertices have the same out- and in-neighborhood. Thus, an important issue in the context of strong products is whether or not two vertices can be distinguished by their neighborhoods. This is captured by the relation S defined on the vertex set of G , which was first introduced by Dörfler and Imrich [2] for undirected graphs.

Let $G = (V, E)$ be a digraph. We define three equivalence relations on V , based on respective neighborhoods. Two vertices $x, y \in V$ are in relation S^+ , in symbols $x \sim_{S^+} y$, if $N_G^+[x] = N_G^+[y]$. Analogously, $x, y \in V$ are in relation S^- if $N_G^-[x] = N_G^-[y]$. Two vertices $x, y \in V$ are in relation S if $x \sim_{S^+} y$ and $x \sim_{S^-} y$. Clearly, S^+ , S^- and S are equivalence relations. For a digraph G let $S^+(v) = \{u \in V(G) \mid u \sim_{S^+} v\}$ denote the equivalence class of S^+ that contains vertex v . Similarly, $S^-(v)$ and $S(v)$ are defined.

We call a digraph $G = (V, E)$ S -thin or thin for short, if for all distinct vertices $x, y \in V$ holds $N_G^+[x] \neq N_G^+[y]$ or $N_G^-[x] \neq N_G^-[y]$. Hence, a digraph is thin, if each equivalence class $S(v)$ of S consists of the single vertex $v \in V(G)$. In other words, G is thin if all vertices can be distinguished by their in- or out-neighborhoods.

The digraph G/S is the usual *quotient graph* with vertex set $V(G/S) = \{a \mid a \text{ is an equivalence class of } S \text{ in } G\}$ and $ab \in E(G/S)$ whenever $xy \in E(G)$ for some $x \in a$ and $y \in b$.

In the following, we give several basic results concerning the relation S and quotients G/S of digraphs G .

Lemma 2.3. *A digraph $G = G_1 \boxtimes G_2$ is thin if and only if G_1 and G_2 are thin.*

Proof. Suppose that G is not thin, and hence there are distinct vertices $x = (x_1, x_2) \in V(G)$ and $y = (y_1, y_2) \in V(G)$ with $N_G^+[(x_1, x_2)] = N_G^+[(y_1, y_2)]$ and $N_G^-[(x_1, x_2)] = N_G^-[(y_1, y_2)]$. This implies that $N_{G_1}^+[x_1] \times N_{G_2}^+[x_2] = N_{G_1}^+[y_1] \times N_{G_2}^+[y_2]$. Hence, $N_{G_1}^+[x_1] = N_{G_1}^+[y_1]$ and $N_{G_2}^+[x_2] = N_{G_2}^+[y_2]$ and since $x \neq y$ we have $x_1 \neq y_1$ or $x_2 \neq y_2$. Similar results hold for the N^- -neighborhoods. Thus if G is not thin, at least one of the factors is not thin.

On the other hand, if G_1 is not thin then $N_{G_1}^+[x_1] = N_{G_1}^+[y_1]$ and $N_{G_1}^-[(x_1, z)] = N_{G_1}^-[(y_1, z)]$ for some $x_1 \neq y_1$ and therefore $N_G^+[(x_1, z)] = N_G^+[(y_1, z)]$ and $N_G^-[(x_1, z)] = N_G^-[(y_1, z)]$ for all $z \in V(G_2)$. \square

Lemma 2.4. *For any digraph $G = (V, E)$ the quotient graph G/S is thin.*

Proof. By definition of the relation S for all $x, x' \in S(v)$ holds $N^+[x] = N^+[x']$ and $N^-[x] = N^-[x']$. Thus, there is an edge $xy \in E$, resp., $yx \in E$ for some $x \in S(v)$ if and only if for all $x' \in S(v)$ holds that $x'y \in E$, resp., $yx' \in E$. Thus, $ab \in E(G/S)$ if and only if for all $x \in a$ and $y \in b$ holds that $xy \in E$.

Assume G/S is not thin. Then, there are distinct vertices $a, b \in V(G/S)$ with $S(a) = S(b)$ and hence, $N_{G/S}^+[a] = N_{G/S}^+[b]$ and $N_{G/S}^-[(a)] = N_{G/S}^-[(b)]$. Hence, $ac \in E(G/S)$ if and only if $bc \in E(G/S)$. By the preceding arguments, it holds that $ac \in E(G/S)$ if and only if for all $x \in a$ and $y \in c$ there is an edge $xy \in E$. Analogously, $bc \in E(G/S)$ if and only if for all $x' \in b$ and $y \in c$ there is an edge $x'y \in E$. Hence, $N_G^+[x] = N_G^+[x']$ for all $x \in a$ and $x' \in b$. By similar arguments one shows that $N_G^-[x] = N_G^-[x']$ for all $x \in a$ and $x' \in b$. But this implies that $a = S(x) = S(x') = b$, a contradiction. \square

Lemma 2.5. *Let G be a digraph. Then the subsets $S^+(v)$, $S^-(v)$ and $S(v)$ induce complete subgraphs for every vertex $v \in V(G)$.*

Proof. If $S^+(v) = \{v\}$, then the assertion is clearly true. Now, let $x, y \in S^+(v)$ be arbitrary. By definition, $y \in N_G^+[y]$ and thus, $y \in N_G^+[x]$ and therefore, $xy \in E(G)$. Analogously, it holds that $x \in N_G^+[y]$ and thus, $yx \in E(G)$. Since this holds for all vertices contained in $S^+(v)$, they induce a complete graph $K_{|S^+(v)|}$. By analogous arguments, the assertion is true for $S^-(v)$. Since $S(v) = S^+(v) \cup S^-(v)$ for all $v \in V(G)$ and since $S^+(v)$ and $S^-(v)$ induce complete graphs, it follows that $S(v)$ induces a complete graph. \square

Lemma 2.6. *For any digraphs G and H holds that $(G \boxtimes H)/S \cong G/S \boxtimes H/S$*

Proof. Reasoning analogously as in the proof for undirected graphs in [7, Lemma 7.2], and by usage of Lemma 2.5 we obtain the desired result. \square

3. Dispensability and the Cartesian Skeleton

A central tool for our PFD algorithms for connected digraphs G is the *Cartesian skeleton* $\mathbb{S}(G)$. The PFD of $\mathbb{S}(G)$ w.r.t. the Cartesian product is utilized to infer the prime factors w.r.t. the strong product of G . This concept was first introduced for undirected graphs by Feigenbaum and Schäffer in [5] and later on improved by Hammack and Imrich, see [6]. Following the illuminating approach of Hammack and Imrich, one removes edges in G that fulfill so-called dispensability conditions, resulting in a subgraph $\mathbb{S}(G)$ that is the desired Cartesian skeleton. In this paper, we provide generalized dispensability conditions and thus, a general definition of the Cartesian skeleton of digraphs. For this purpose we first give the definitions of the so-called (weak) N^+ -condition and N^- -condition. Based on this, we will provide a general concept of dispensability for digraphs, which in turn enables us to define the Cartesian skeleton $\mathbb{S}(G)$. We prove that $\mathbb{S}(G)$ is a connected spanning subgraph, provided G is connected. Moreover for S -thin digraphs the Cartesian skeleton is uniquely determined and we obtain $\mathbb{S}(H \boxtimes K) \cong \mathbb{S}(H) \boxtimes \mathbb{S}(K)$.

Definition 3.1. *Let G be a digraph and $xy \in E(G)$, $z \in V(G)$ be an arbitrary edge, resp, vertex of G . We say xy satisfies the N^+ -condition with z if one of the following conditions is fulfilled:*

- (1⁺) $N_G^+[x] \subset N_G^+[z] \subset N_G^+[y]$
- (2⁺) $N_G^+[y] \subset N_G^+[z] \subset N_G^+[x]$
- (3⁺) $N_G^+[x] \cap N_G^+[y] \subset N_G^+[x] \cap N_G^+[z]$ and $N_G^+[x] \cap N_G^+[y] \subset N_G^+[y] \cap N_G^+[z]$

We say xy satisfies the weak N^+ -condition with z , if the following condition is fulfilled:

$$N_G^+[x] \cap N_G^+[y] \subseteq N_G^+[x] \cap N_G^+[z] \text{ and } N_G^+[x] \cap N_G^+[y] \subseteq N_G^+[y] \cap N_G^+[z]$$

Analogously, by replacing “ N_G^+ ” by “ N_G^- ” we get Conditions (1⁻), (2⁻), (3⁻), for the definition of the N^- -condition with z , respectively, for the definition of the weak N^- -condition with z .

Definition 3.2. *Let G be a digraph. An edge $xy \in E(G)$ is dispensable if at least one of the following conditions is satisfied:*

- (D1) *There exists a vertex $z \in V(G)$ such that xy satisfies the N^+ - and N^- -condition with z .*
- (D2) *There are vertices $z_1, z_2 \in V(G)$ such that both conditions holds:*
 - (a) *xy satisfies (3⁺) of the N^+ -condition with z_1 and the weak N^- -condition with z_1 .*
 - (b) *xy satisfies (3⁻) of the N^- -condition with z_2 and the weak N^+ -condition with z_2 .*
- (D3) *There exists a vertex $z \in V(G)$ such that xy satisfies the N^+ -condition with z and at least one of the following holds: $N^-[x] = N^-[z]$ or $N^-[y] = N^-[z]$.*
- (D4) *There exists a vertex $z \in V(G)$ such that xy satisfies the N^- -condition with z and at least one of the following holds: $N^+[x] = N^+[z]$ or $N^+[y] = N^+[z]$.*

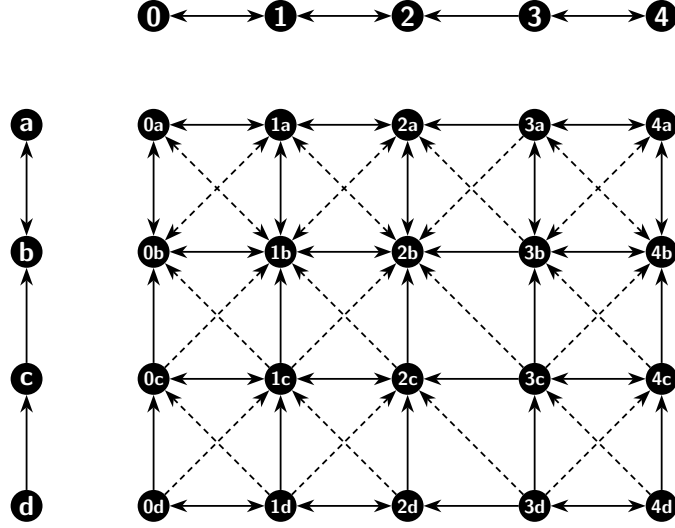


Figure 1: Shown is the strong product of two thin digraphs G_1 and G_2 . The dashed edges are dispensable and thus, $\mathbb{S}(G_1 \boxtimes G_2) = \mathbb{S}(G_1) \square \mathbb{S}(G_2)$ is the subgraph that contains all non-dashed edges. By way of example, the edge $(0d)(1c)$ satisfies (D1) with $z = (1d)$; the edge $(2d)(1c)$ satisfies (D2) with $z_1 = (1d), z_2 = (2c)$; the edge $(3c)(4b)$ satisfies (D3) with $z = (3b)$; the edge $(3a)(2b)$ satisfies (D4) with $z = (3b)$; and the edge $(4a)(3b)$ satisfies (D5) with $z_1 = (4b), z_2 = (3a)$.

(D5) There are distinct vertices $z_1, z_2 \in V(G)$, both distinct from x and y , such that $N^+[x] = N^+[z_1]$, $N^-[x] = N^-[z_2]$, $N^-[z_1] = N^-[y]$ and $N^+[z_2] = N^+[y]$.

All other edges in $E(G)$ are non-dispensable.

Note, if one considers undirected graphs $G = (V, E)$ as graphs for which $N^+[v] = N^-[v]$ for all $v \in V$, then none of the Conditions (D2)-(D4) can be fulfilled for G . Moreover if this undirected graph is thin, then Condition (D5) cannot be satisfied. In other words, the definition of dispensability reduces to (D1) and thus, coincides with that for undirected graphs given by Hammack and Imrich [6].

Remark 1. Let $G = (V, E)$ be a digraph and assume the edge $xy \in E$ is dispensable by one of the Conditions (D1), (D3) (D4) with some vertex $z \in V$ or (D2), (D5) with some $z_1, z_2 \in V$. It is now an easy task to verify that $z \in N^+[x] \cup N^-[x]$ and $z \in N^+[y] \cup N^-[y]$. The same is true for z_1 and z_2 .

We are now in the position to define the Cartesian skeleton of digraphs.

Definition 3.3. The Cartesian skeleton of a digraph G is the digraph $\mathbb{S}(G)$ that is obtained from G by removing all dispensable edges. More precise, the Cartesian skeleton $\mathbb{S}(G)$ has vertex set $V(G)$ and edge set $E(\mathbb{S}(G)) = E(G) \setminus D(G)$, where $D(G)$ denotes the set of dispensable edges in G .

In the following, we will show that non-Cartesian edges are dispensable and moreover that $\mathbb{S}(H \boxtimes K) = \mathbb{S}(H) \square \mathbb{S}(K)$, whenever H and K are thin graphs.

Lemma 3.4. Let $G = H \boxtimes K$ be a thin digraph. Then every non-Cartesian edge is dispensable and thus, every edge of $\mathbb{S}(G)$ is Cartesian w.r.t. this factorization.

Proof. Suppose that the edge $(h, k)(h', k') \in E(G)$ is non-Cartesian. We have to examine several cases.

Assume $N_H^+[h] \neq N_H^+[h']$ and $N_K^+[k] \neq N_K^+[k']$. Then

$$\begin{aligned} N_G^+[(h, k)] \cap N_G^+[(h', k')] &= (N_H^+[h] \cap N_H^+[h']) \times (N_K^+[k] \cap N_K^+[k']) \\ &\subseteq N_H^+[h] \times (N_K^+[k] \cap N_K^+[k']) \\ &= N_G^+[(h, k)] \cap N_G^+[(h, k')] \end{aligned} \tag{1}$$

$$\begin{aligned}
N_G^+[(h, k)] \cap N_G^+[(h', k')] &= (N_H^+[h] \cap N_H^+[h']) \times (N_K^+[k] \cap N_K^+[k']) \\
&\subseteq (N_H^+[h] \cap N_H^+[h']) \times N_K^+[k'] \\
&= N_G^+[(h, k')] \cap N_G^+[(h', k')]
\end{aligned} \tag{2}$$

Interchanging the roles of h and k with h' and k' gives us by similar arguments:

$$N_G^+[(h', k')] \cap N_G^+[(h, k)] \subseteq N_G^+[(h', k')] \cap N_G^+[(h', k)] \text{ and} \tag{3}$$

$$N_G^+[(h', k')] \cap N_G^+[(h, k)] \subseteq N_G^+[(h', k)] \cap N_G^+[(h, k)]. \tag{4}$$

Notice that $N_G^+[(h, k)] \cap N_G^+[(h', k')] \neq \emptyset$, since $(h, k)(h', k') \in E(G)$ implies that $(h', k') \in N_G^+[(h, k)] \cap N_G^+[(h', k)]$. The following four cases can occur:

1. All inclusions in Eq. (1) - (4) are inequalities, thus $(h, k)(h', k')$ satisfies (3^+) of the N^+ -condition with z by choosing $z = (h, k')$ or $z = (h', k)$.
2. Only the first two inclusions (Eq. (1) - (2)) are inequalities, thus $(h, k)(h', k')$ satisfies (3^+) of the N^+ -condition with $z = (h, k')$ and the weak N^+ -condition with $z = (h', k)$.
3. Symmetrically, if only the last two inclusions (Eq. (3) - (4)) are inequalities, then $(h, k)(h', k')$ satisfies (3^+) of the N^+ -condition with $z = (h', k)$ and the weak N^+ -condition with $z = (h, k')$.
4. At least one of the first two and one of last two inclusions are equality. From the first two formulas we get $N_H^+[h] \cap N_H^+[h'] = N_H^+[h]$ or $N_K^+[k] \cap N_K^+[k'] = N_K^+[k']$. Due to the assumption $N_K^+[h] \neq N_K^+[h']$ and $N_K^+[k] \neq N_K^+[k']$ this implies

$$N_H^+[h] \subset N_H^+[h'] \text{ or } N_K^+[k'] \subset N_K^+[k].$$

Similarly we get from the last two formulas

$$N_H^+[h'] \subset N_H^+[h] \text{ or } N_K^+[k] \subset N_K^+[k'].$$

This implies we have

$$N_H^+[h] \subset N_H^+[h'] \text{ and } N_K^+[k] \subset N_K^+[k'] \text{ or } N_H^+[h'] \subset N_H^+[h] \text{ and } N_K^+[k'] \subset N_K^+[k]$$

and thus

$$N_G^+[(h, k)] \subset N_G^+[(h, k')] \subset N_G^+[(h', k')] \text{ and } N_G^+[(h, k)] \subset N_G^+[(h', k)] \subset N_G^+[(h', k')]$$

or

$$N_G^+[(h', k')] \subset N_G^+[(h, k')] \subset N_G^+[(h, k)] \text{ and } N_G^+[(h', k')] \subset N_G^+[(h', k)] \subset N_G^+[(h, k)].$$

Therefore, also in this case $(h, k)(h', k')$ satisfies the N^+ -condition with $z = (h, k')$ and with $z = (h', k)$.

So far we treated the N^+ -neighborhoods under the assumption that $N_H^+[h] \neq N_H^+[h']$ and $N_K^+[k] \neq N_K^+[k']$. For the N^- -neighborhoods the situation can be treated analogously, if we assume that $N_H^-[h] \neq N_H^-[h']$ and $N_K^-[k] \neq N_K^-[k']$. Then, we obtain the same latter four cases just by replacing N_H^+ and N_K^+ , by N_H^- and N_K^- , respectively. Now, it is easy to verify that every combination of the Cases 1. - 4. for N^+ - and N^- -neighborhoods leads to one of the conditions (D1) or (D2).

Assume that $N_H^+[h] = N_H^+[h']$ and $N_K^-[k] = N_K^-[k']$. Then Condition (D5) holds for the edge $(h, k)(h', k')$ with $z_1 = (h', k)$ and $z_2 = (h, k')$. Analogous arguments show that Condition (D5) is satisfied, if $N_H^-[h] = N_H^-[h']$ and $N_K^+[k] = N_K^+[k']$.

Finally, assume that $N_H^+[h] = N_H^+[h']$ and $N_K^-[k] \neq N_K^-[k']$. By thinness it must hold $N_H^-[h] \neq N_H^-[h']$. Thus, we have the Cases 1. - 4. for N^- -neighborhoods. In particular, for all four cases we can infer that the edge $(h, k)(h', k')$ satisfies the N^- -condition with vertex (h, k') or (h', k) . Hence, Condition (D4) is satisfied since $N_G^+[(h, k)] = N_G^+[(h', k)]$ and $N_G^+[(h, k')] = N_G^+[(h', k')]$. If $N_H^+[h] \neq N_H^+[h']$ and $N_K^-[k] = N_K^-[k']$ then we obtain by similar arguments, that (D3) is satisfied.

Hence, in all cases we can observe that non-Cartesian edges fulfill one of the Condition (D1) – (D5) and are thus, dispensable. \square

Lemma 3.5. *If H, K are thin digraphs, then $\mathbb{S}(H \boxtimes K) \subseteq \mathbb{S}(H) \square \mathbb{S}(K)$.*

Proof. In the following, we will denote in some cases for simplicity the product $H \boxtimes K$ by G . By Lemma 3.4, the subgraph $\mathbb{S}(H \boxtimes K)$ contains Cartesian edges only. Hence, by commutativity of the Cartesian product, it remains to show that for every non-dispensable Cartesian edge $(h, k)(h', k)$ contained in $\mathbb{S}(H \boxtimes K)$, there is an edge $hh' \in \mathbb{S}(H)$ and thus $(h, k)(h', k)$ is also contained in $\mathbb{S}(H) \square \mathbb{S}(K)$.

By contraposition, assume that hh' is dispensable in H , that is, one of the Conditions (D1)-(D5) is fulfilled.

Assume (D1) holds for hh' with some $z \in V(H)$. Then one of the following conditions holds $(1^+) N_H^+[h] \subset N_H^+[z] \subset N_H^+[h']$, $(2^+) N_H^+[h'] \subset N_H^+[z] \subset N_H^+[h]$ or $(3^+) N_H^+[h] \cap N_H^+[h'] \subset N_H^+[h] \cap N_H^+[z]$ and $N_H^+[h] \cap N_H^+[h'] \subset N_H^+[h'] \cap N_H^+[z]$. If we multiply every neighborhood in the inclusions with $N_K^+[k]$ we get a N^+ -condition for $(h, k)(h', k)$ with (z, k) . Analogously, if hh' satisfies the N^- -condition with $z \in V(H)$, then $(h, k)(h', k)$ satisfies N^- -condition with (z, k) . Thus Condition (D1) for hh' implies (D1) for $(h, k)(h', k)$.

Assume (D2) holds for hh' . Hence there are vertices $z_1, z_2 \in V(H)$ s.t. hh' satisfies (3^+) of the N^+ -condition with z_1 and the weak N^- -condition with z_1 , as well as, the (3^-) of the N^- -condition with z_2 and the weak N^+ -condition with z_2 . As argued before, the edge $(h, k)(h', k)$ satisfies (3^+) of the N^+ -condition with (z_1, k) and (3^-) of the N^- -condition with (z_2, k) . For hh' and the weak N^- -condition holds $N_H^-[h] \cap N_H^-[h'] \subseteq N_H^-[h] \cap N_H^-[z_1]$ and $N_H^-[h] \cap N_H^-[h'] \subseteq N_H^-[h'] \cap N_H^-[z_1]$. Again, if we multiply every inclusion with $N^-[k]$ we can infer that

$$N_G^-[h, k] \cap N_G^-[h', k] \subseteq N_G^-[h, k] \cap N_G^-[z_1, k]$$

and

$$N_G^-[h, k] \cap N_G^-[h', k] \subseteq N_G^-[h', k] \cap N_G^-[z_1, k].$$

Thus Item (a) of Condition (D2) is satisfied for $(h, k)(h', k)$ with (z_1, k) . By analogous arguments, we derive that Item (b) of Condition (D2) is satisfied for $(h, k)(h', k)$ with (z_2, k) . Hence, Condition (D2) for hh' implies that (D2) holds for $(h, k)(h', k)$.

For Condition (D3), resp., (D4) we can infer by the preceding arguments, that the N^+ -condition, resp., N^- -condition for $(h, k)(h', k)$ with (z, k) is fulfilled, whenever these conditions are satisfied for hh' with z . Now, $N_H^-[h] = N_H^-[z]$ or $N_H^-[h'] = N_H^-[z]$ implies $N_G^-[h, k] = N_G^-[z, k]$ or $N_G^-[h', k] = N_G^-[z, k]$ and similarly this holds for N^+ -neighborhoods. Hence (D3), resp., (D4) are fulfilled for the edge $(h, k)(h', k)$.

Finally, consider Condition (D5). Assume there are distinct vertices $z_1, z_2 \in V(G)$ such that $N_H^+[h] = N_H^+[z_1]$, $N_H^-[h] = N_H^-[z_2]$, $N_H^-[z_1] = N_H^-[h']$ and $N_H^+[z_2] = N_H^+[h']$. This implies that $N_G^+[(h, k)] = N_G^+[(z_1, k)]$, $N_G^-[(h, k)] = N_G^-[(z_2, k)]$, $N_G^-[(z_1, k)] = N_G^-[(h', k)]$ and $N_G^+[(z_2, k)] = N_G^+[(h', k)]$ and therefore, Condition (D5) is fulfilled for the edge $(h, k)(h', k)$.

To summarize, if hh' is dispensable then $(h, k)(h', k)$ is dispensable and hence, $\mathbb{S}(H \boxtimes K) \subseteq \mathbb{S}(H) \square \mathbb{S}(K)$. \square

Proposition 3.6. *If H, K are thin graphs, then $\mathbb{S}(H \boxtimes K) = \mathbb{S}(H) \square \mathbb{S}(K)$.*

Proof. By Lemma 3.5, it remains to prove that $\mathbb{S}(H) \square \mathbb{S}(K) \subseteq \mathbb{S}(H \boxtimes K)$. Moreover, by commutativity of the products, we must only show that for every edge $(h, k)(h', k) \in E(\mathbb{S}(H) \square \mathbb{S}(K))$ holds that $(h, k)(h', k)$ is not dispensable in $H \boxtimes K$.

For contraposition, assume $(h, k)(h', k)$ is dispensable in $H \boxtimes K$. We will prove that then hh' is dispensable in H . In the following, we will denote in some cases for simplicity the product $H \boxtimes K$ by G .

Let us assume that Condition (D1) holds for $(h, k)(h', k)$ with $z = (z', z'')$. If Condition (1^+) is fulfilled then $N_G^+[(h, k)] \subset N_G^+[(z', z'')] \subset N_G^+[(h', k)]$ and we get

$$N_H^+[h] \times N_K^+[k] \subset N_H^+[z'] \times N_K^+[z''] \subset N_H^+[h'] \times N_K^+[k].$$

The latter implies that $N_K^+[z''] = N_K^+[k]$, which causes $N_H^+[h] \subset N_H^+[z'] \subset N_H^+[h']$ and hence, (1^+) is fulfilled in H for hh' with z' . If Condition (2^+) is fulfilled, then analogous arguments show that $N_H^+[h'] \subset N_H^+[z'] \subset N_H^+[h]$. Assume now that Condition (3^+) holds: $N_G^+[(h, k)] \cap N_G^+[(h', k)] \subset N_G^+[(h, k)] \cap N_G^+[(z', z'')]$ and $N_G^+[(h, k)] \cap N_G^+[(h', k)] \subset N_G^+[(h', k)] \cap N_G^+[(z', z'')]$. Therefore,

$$(N_H^+[h] \cap N_H^+[h']) \times N_K^+[k] \subset (N_H^+[h] \cap N_H^+[z']) \times (N_K^+[k] \cap N_K^+[z''])$$

and

$$(N_H^+[h] \cap N_H^+[h']) \times N_K^+[k] \subset (N_H^+[h'] \cap N_H^+[z']) \times (N_K^+[k] \cap N_K^+[z'']).$$

Since $hh' \in E(H)$, we can conclude that $N_H^+[h] \cap N_H^+[h'] \neq \emptyset$. Hence, the latter implies that $N_K^+[k] \subseteq N_K^+[k] \cap N_K^+[z'']$ and thus, $N_K^+[k] = N_K^+[k] \cap N_K^+[z'']$. Then it must hold that $N_H^+[h] \cap N_H^+[h'] \subset N_H^+[h] \cap N_H^+[z']$ and $N_H^+[h] \cap N_H^+[h'] \subset N_H^+[h'] \cap N_H^+[z']$, which yields (3^+) for hh' with z' . Similarly all N^- -conditions can be transferred from $(h, k)(h', k)$ with (z', z'') to hh' with z' . Hence whenever Condition (D1) is fulfilled for $(h, k)(h', k)$ with $z = (z', z'')$ then (D1) holds for hh' with z' , as well.

Now, assume that Condition (D2) holds for $(h, k)(h', k)$ with $z_1 = (z'_1, z'_1)$ and $z_2 = (z'_2, z'_2)$. By the above arguments it is clear that (3^+) is fulfilled for hh' with z'_1 and (3^-) is fulfilled for hh' with z'_2 . Consider the weak N^- -condition for $(h, k)(h', k)$ with z_1 :

$$N_G^-(h, k) \cap N_G^-(h', k) \subseteq N_G^-(h, k) \cap N_G^-(z'_1, z'_1)$$

and

$$N_G^-(h, k) \cap N_G^-(h', k) \subseteq N_G^-(h', k) \cap N_G^-(z'_1, z'_1).$$

Obviously this implies $N_H^-[h] \cap N_H^-[h'] \subseteq N_H^-[h] \cap N_H^-[z'_1]$ and $N_H^-[h] \cap N_H^-[h'] \subseteq N_H^-[h'] \cap N_H^-[z'_1]$. Therefore, the weak N^- -condition holds for hh' with z'_1 . By analogous arguments, we obtain that also the weak N^+ -condition is fulfilled for hh' with z'_2 . Hence, Condition (D2) holds for hh' with z'_1 and z'_2 .

If Condition (D3) is fulfilled for $(h, k)(h', k)$ with $z = (z', z'')$ then by the above arguments, the N^- -condition holds for hh' with z' . Moreover, it holds $N_G^-(h, k) = N_G^-(z', z'')$ or $N_G^-(h', k) = N_G^-(z', z'')$, but this is only possible if $N_H^-[h] = N_H^-[z']$ or $N_H^-[h'] = N_H^-[z']$. Hence, (D3) holds for hh' with z' . By analogous arguments we can infer that Condition (D4) holds for hh' with z' whenever (D4) holds for $(h, k)(h', k)$ with $z = (z', z'')$.

It remains to check Condition (D5). Let (z'_1, z'_1) and (z'_2, z'_2) be two distinct vertices such that $N_G^+(h, k) = N_G^+(z'_1, z'_1)$, $N_G^-(h, k) = N_G^-(z'_2, z'_2)$, $N_G^-(z'_1, z'_1) = N_G^-(h', k)$ and $N_G^+(z'_2, z'_2) = N_G^+(h', k)$. Again, this is only possible if $N_H^+[h] = N_H^+[z'_1]$, $N_H^-[h] = N_H^-[z'_2]$, $N_H^+[z'_1] = N_H^+[h']$ and $N_H^-[z'_2] = N_H^-[h']$. Clearly, since $(h, k)(h', k) \in E(G)$ the vertices h and h' are distinct. However, we must also verify that $z'_1 \neq z'_2$ and $z'_1, z'_2 \notin \{h, h'\}$.

Assume $z'_1 = h$. Since by assumption, $N_G^-(z'_1, z'_1) = N_G^-(h', k)$ it must hold $N_K^-[z'_1] = N_K^-[k]$. Then we can infer that $N_H^-[h] \times N_K^-[k] = N_H^-[h] \times N_K^-[z'_1] = N_H^-[z'_1] \times N_K^-[z'_1]$ and thus, $N_G^-(h, k) = N_G^-(z'_1, z'_1)$. However, this contradicts the fact that G is thin, since we assumed that $N_G^+(h, k) = N_G^+(z'_1, z'_1)$. Using analogous arguments one shows that $z'_1, z'_2 \notin \{h, h'\}$.

Finally, assume that $z'_1 = z'_2$. First, note that $N_G^-(z'_1, z'_1) = N_G^-(h', k)$ implies that $N_K^-[z'_1] = N_K^-[k]$. Second, $N_G^-(h, k) = N_G^-(z'_2, z'_2)$ implies that $N_H^-[h] = N_H^-[z'_2]$ and thus, $N_H^-[h] = N_H^-[z'_1]$. Therefore, by the same arguments as before, we obtain that $N_G^-(h, k) = N_G^-(z'_1, z'_1)$, which contradicts that G is thin, since by assumption $N_G^+(h, k) = N_G^+(z'_1, z'_1)$. Hence, Condition (D5) is fulfilled for hh' with z'_1 and z'_2 .

To summarize, dispensability of $(h, k)(h', k)$ in $H \boxtimes K$ implies dispensability of hh' in H . By commutativity of the products, we can conclude that $\mathbb{S}(H) \square \mathbb{S}(K) \subseteq \mathbb{S}(H \boxtimes K)$, that together with Lemma 3.5 implies $\mathbb{S}(H \boxtimes K) = \mathbb{S}(H) \square \mathbb{S}(K)$ \square

In the following, we will show that the Cartesian skeleton $\mathbb{S}(G)$ of a connected thin digraph G is connected.

Lemma 3.7. *Let $G = (V, E)$ be a thin connected digraph and let $S^+(v)$ and $S^-(v)$ be the corresponding S^+ - and S^- -classes containing vertex $v \in V$. Then all vertices contained in $S^+(v)$ lie in the same connected component of $\mathbb{S}(G)$, i.e., there is always a walk consisting of non-dispensable edges only, that connects all vertices $x, y \in S^+(v)$. The same is true for all vertices contained in $S^-(v)$.*

Proof. If $|S^+(v)| = 1$ there is nothing to show. Thus, assume $x, y \in S^+(v)$. By Lemma 2.5 there is an edge $xy \in E(G)$. Assume that this edge xy is dispensable. Since $N^+[x] = N^+[y]$, none of the Conditions (D1), (D2), and (D3) can be satisfied for the edge xy . Moreover, (D5) can not hold, since otherwise we would have $N^+[x] = N^+[z_1] = N^+[y] = N^+[z_2]$ and $N^-[x] = N^-[z_2]$ and thus, G would not be thin. Therefore, if xy is dispensable, then Condition (D4) must hold. Thus, there is a vertex z such that one of the N^- -conditions (1^-) , (2^-) or (3^-) with z holds for xy and $N^+[x] = N^+[z]$ or $N^+[z] = N^+[y]$. Since $N^+[x] = N^+[y]$, we can conclude that z must be contained in $S^+(v)$.

First assume that Condition (1^-) for xy with z is satisfied and therefore in particular, $N^-[x] \subset N^-[y]$. Consider the maximal chain $N^-[x] \subset N^-[z_1] \subset \dots \subset N^-[z_{k-1}] \subset N^-[y]$ of neighborhoods between $N^-[x]$ and $N^-[y]$ ordered

by proper inclusions, where $z_i \in S^+(v)$ for all $i \in \{1, \dots, k-1\}$. To simplify the notation let $x = z_0$ and $y = z_k$. Lemma 2.5 implies that $z_i z_{i+1} \in E(G)$ for all $i \in \{0, \dots, k-1\}$. We show that the edges $z_i z_{i+1}$ for all $i \in \{0, \dots, k-1\}$ are non-dispensable. By the preceding arguments, such an edge $z_i z_{i+1}$ can only be dispensable if Condition (D4) is satisfied, and thus in particular, if there exists a vertex $z' \in S^+(v)$ such that the N^- -condition for $z_i z_{i+1}$ holds with z' . Since $N^-[z_i] \subset N^-[z_{i+1}]$ we can conclude that Condition (2 $^-$) cannot be satisfied. Moreover, $N^-[z_i] \cap N^-[z_{i+1}] \subset N^-[z_i] \cap N^-[z]$ is not possible, and thus, Condition (3 $^-$) cannot be satisfied. Furthermore, since we constructed a maximal chain of proper included neighborhoods, $N^-[z_i] \subset N^-[z'] \subset N^-[z_{i+1}]$ is not possible and and therefore, Condition (1 $^-$) cannot be satisfied. Hence, none of the N^- -conditions for the edges $xz_1, z_1 z_2, \dots, z_k y$ can be satisfied, which yields a walk in $\mathbb{S}(G)$ connecting x and y . Therefore, all $x, y \in S^+(v)$ with $N^-[x] \subset N^-[y]$ lie in the same connected component of $\mathbb{S}(G)$. By analogous arguments one shows that x and y lie in the same connected component of $\mathbb{S}(G)$ if Condition (2 $^-$) and thus, $N^-[y] \subset N^-[x]$ is satisfied.

We summarize at this point: All $x, y \in S^+(v)$ where the edge xy fulfill Condition (1 $^-$) or (2 $^-$) are in the same connected component of $\mathbb{S}(G)$. Now assume for contradiction, that there are vertices $x, y \in S^+(v)$ (and hence, an edge $xy \in E(G)$) that are in different connected components of $\mathbb{S}(G)$. This is only possible if the edge xy is dispensable by Condition (3 $^-$) and thus if $N^-[x] \cap N^-[y] \subset N^-[x] \cap N^-[z]$ and $N^-[x] \cap N^-[y] \subset N^-[y] \cap N^-[z]$. Define for arbitrary vertices $x, y \in S^+(v)$ the integer $k_{xy} = |N^-[x] \cap N^-[y]|$ and take among all $x, y \in S^+(v)$ that are in different connected components of $\mathbb{S}(G)$ the ones that have largest value k_{xy} . Note, $k_{xz}, k_{yz} > k_{xy}$. Moreover, since $z \in S^+(v)$ and we have taken $x, y \in S^+(v)$ that have largest integer k_{xy} among all vertices that are in different connected components of $\mathbb{S}(G)$, we can conclude that x and z , as well as y and z are in the same connected component in $\mathbb{S}(G)$, a contradiction. This completes the proof for the case $x, y \in S^+(v)$.

By analogous arguments one shows that the statement is true for $S^-(v)$. \square

Lemma 3.8. *Let $G = (V, E)$ be a thin connected digraph and $x, y \in V$ with $N^+[x] \subset N^+[y]$ or $N^-[x] \subset N^-[y]$. Then there is a walk in $\mathbb{S}(G)$ connecting x and y .*

Proof. Assume first that $N^+[x] \subset N^+[y]$. Note, one can always construct a maximal chain of vertices with $N^+[x] \subset N^+[z_1] \subset \dots \subset N^+[y]$ and connect walks inductively, whenever the statement is true. Therefore, we can assume that $N^+[x] \subset N^+[y]$ with no $z \in V$ such that $N^+[x] \subset N^+[z] \subset N^+[y]$. Clearly, $N^+[x] \subset N^+[y]$ implies $xy \in E(G)$. Assume xy is dispensable. Since by assumption there is no z with $N^+[x] \subset N^+[z] \subset N^+[y]$ it follows that Condition (1 $^+$) can not hold. Moreover, since $N^+[x] \subset N^+[y]$ Condition (2 $^+$) can not hold. The latter also implies $N^+[x] \cap N^+[y] = N^+[x]$ and thus Condition (3 $^+$) can not be fulfilled, since $N^+[x] = N^+[x] \cap N^+[y] \subset N^+[x] \cap N^+[z]$ is not possible. Thus xy does not satisfy the N^+ -condition and thus it cannot be dispensable by Conditions (D1), (D2) or (D3).

If it is dispensable by Condition (D5), then there exists a vertex z_1 with $N^+[x] = N^+[z_1]$ and $N^-[z_1] = N^-[y]$. Hence, $z_1 \in S^+(x)$ and $z_1 \in S^-(y)$. By Lemma 3.7 there is a x, z_1 -walk and z_1, y -walk and thus, a walk connecting x and y consisting of non-dispensable edges only. This together with Lemma 3.7 implies that, also all vertices $x' \in S^+(x)$ and $y' \in S^+(y)$ are connected by a walk of non-dispensable edges.

Assume now for contradiction that vertices x and y are in different connected components of $\mathbb{S}(G)$. By Lemma 3.7, all vertices contained $S^+(x)$ are in same connected component of $\mathbb{S}(G)$. The same is true for all vertices contained in $S^+(y)$. Hence if x and y are in different components then all vertices contained $S^+(x)$ must be in a different connected component of $\mathbb{S}(G)$ than the vertices contained in $S^+(y)$. By the preceding arguments, this can only happen, when all edges $x'y' \in E$ with $x' \in S^+(x)$ and $y' \in S^+(y)$ are dispensable by Condition (D4). We examine now three cases: there are $x' \in S^+(x)$ and $y' \in S^+(y)$ with (i) $N^-[x'] \subset N^-[y']$, (ii) $N^-[y'] \subset N^-[x']$ or (iii) none of the cases (i), (ii) hold.

Case (i) $N^-[x'] \subset N^-[y']$: W.l.o.g. assume that x' and y' are chosen, such that $|N^-[y'] - N^-[x']|$ becomes minimal among all such pairs $x' \in S^+(x)$ and $y' \in S^+(y)$. Since the edge $x'y'$ must be dispensable by Condition (D4), there is a vertex $z \in V$ with $N^+[x'] = N^+[z]$ or $N^+[y'] = N^+[z]$ and xy satisfies the N^- -condition with z . Clearly, Condition (2 $^-$) with $N^-[y'] \subset N^-[z] \subset N^-[x']$ cannot be fulfilled. Moreover, Condition (3 $^-$) cannot hold, since $N^-[x'] \subset N^-[y']$ implies that $N^-[x'] = N^-[x'] \cap N^-[y']$ and thus $N^-[x'] \cap N^-[y'] \subset N^-[x'] \cap N^-[z]$ is not possible. Thus, assume $x'y'$ fulfills Condition (1 $^-$) with z , then $N^-[x'] \subset N^-[z] \subset N^-[y']$. Since $N^+[x'] = N^+[z]$ or $N^+[y'] = N^+[z]$ we have that $z \in S^+(x)$ or $z \in S^+(y)$. But then $|N^-[z] - N^-[x']|$ or $|N^-[y'] - N^-[z]|$ is smaller than $|N^-[y'] - N^-[x']|$, a contradiction. Hence, $x'y'$ is not dispensable and thus the vertices in $S^+(x)$ and $S^+(y)$ cannot be in different connected components of $\mathbb{S}(G)$.

Case (ii) $N^-[y'] \subset N^-[x']$: By analogous arguments as in Case (i) one shows that the edge $x'y'$ connects $S^+(x)$ and $S^+(y)$ when x' and y' are chosen such that $|N^-[y'] - N^-[x']|$ becomes minimal.

Case (iii): Assume that neither Case (i) nor (ii) holds. Hence, if $x'y'$ with $x' \in S^+(x)$ and $y' \in S^+(y)$ is dispensable by Condition (D4), then neither (1^-) nor (2^-) of the N^- -condition can be fulfilled. Hence, (3^-) with some vertex z must hold, that is, $N^-[x'] \cap N^-[y'] \subset N^-[z] \cap N^-[y']$ and $N^-[x'] \cap N^-[y'] \subset N^-[z] \cap N^-[x']$. W.l.o.g. assume that x' and y' are chosen, such that $|N^-[x'] \cap N^-[y']|$ becomes maximal among all such pairs $x' \in S^+(x)$ and $y' \in S^+(y)$. By Condition (D4) it holds that $N^+[x'] = N^+[z]$ or $N^+[y] = N^+[z]$ and thus, $z \in S^+(x)$ or $z \in S^+(y)$. However, Condition (3^-) is fulfilled, and thus $|N^-[x'] \cap N^-[z]|$ and $|N^-[z] \cap N^-[y']|$ are greater than $|N^-[x'] \cap N^-[y']|$, a contradiction to the choice of x' and y' . Hence, $x'y'$ is *not* dispensable and thus, the vertices contained $S^+(x)$ and $S^+(y)$ cannot lie in different connected components of $\mathbb{S}(G)$.

By analogous arguments one shows, that $x, y \in V$ are in the same connected component of $\mathbb{S}(G)$ if $N^-[x] \subset N^-[y]$. \square

Proposition 3.9. *If $G = (V, E)$ is thin and connected, then $\mathbb{S}(G)$ is connected.*

Proof. For each edge $xy \in E(G)$ define an integer

$$k_{xy} = |N^+[x] \cap N^+[y]| + |N^-[x] \cap N^-[y]|.$$

Assume for contradiction, that x and y are in different connected components of $\mathbb{S}(G)$. Hence, xy must be dispensable. Take among all dispensable edges $xy \in E$, where x and y are in different components of $\mathbb{S}(G)$ the ones that have largest value k_{xy} .

By the same arguments as in the proof of Lemma 3.8 the edge xy cannot be dispensable by Condition (D5), since then there is a vertex $z_1 \in S^+(x)$ and $z_1 \in S^-(y)$ and by Lemma 3.7, there is a walk connecting x and y consisting of non-dispensable edges only and thus x and y are in the same connected component of $\mathbb{S}(G)$.

Moreover, if for x and y one of the Conditions (1^+) , (2^+) , (1^-) or (2^-) holds, then Lemma 3.8 implies that x and y are in the same connected component of $\mathbb{S}(G)$.

If (D1) with (3^+) and (3^-) is satisfied, then $N^+[x] \cap N^+[y] \subset N^+[x] \cap N^+[z]$, $N^-[x] \cap N^-[y] \subset N^-[x] \cap N^-[z]$, $N^+[x] \cap N^+[y] \subset N^+[y] \cap N^+[z]$ and $N^-[x] \cap N^-[y] \subset N^-[y] \cap N^-[z]$. Note, by Remark 1 there is an edge $xz \in E$ or $zx \in E$, as well as, an edge $yz \in E$ or $zy \in E$. But then, $k_{xz} > k_{xy}$ and $k_{yz} > k_{xy}$. Since xy is chosen among all dispensable edges where x and y are in different components that have maximal value k_{xy} we can conclude that x and z , resp., y and z are in the same connected component of $\mathbb{S}(G)$ or that xz , resp., yz are non-dispensable. Both cases lead to a contradiction, since then x and y would be connected by a walk in $\mathbb{S}(G)$.

If (D2) holds, then in particular Condition (3^+) and the weak N^- -condition holds with z_1 . Therefore, $N^+[x] \cap N^+[y] \subset N^+[x] \cap N^+[z_1]$, $N^+[x] \cap N^+[y] \subset N^+[y] \cap N^+[z_1]$, $N^-[x] \cap N^-[y] \subseteq N^-[x] \cap N^-[z_1]$ and $N^-[x] \cap N^-[y] \subseteq N^-[y] \cap N^-[z_1]$. By Remark 1 there is an edge $xz_1 \in E$ or $z_1x \in E$, as well as, an edge $yz_1 \in E$ or $z_1y \in E$. Again, $k_{xz_1} > k_{xy}$ and $k_{z_1y} > k_{xy}$. By analogous arguments as in the latter case, we obtain a contradiction.

If (D3) holds, then there is a vertex $z \in V$ with $N^+[x] \cap N^+[y] \subset N^+[x] \cap N^+[z]$, $N^+[x] \cap N^+[y] \subset N^+[y] \cap N^+[z]$ and $N^-[x] = N^-[z]$ or $N^-[y] = N^-[z]$. If $N^-[x] = N^-[z]$, then Lemma 3.7 implies that x and z are connected by a walk. Moreover, for zy holds then $N^-[x] \cap N^-[y] = N^-[y] \cap N^-[z]$ and still $N^+[x] \cap N^+[y] \subset N^+[y] \cap N^+[z]$. Note, by Remark 1 there is an edge $yz \in E$ or $zy \in E$. Again, $k_{yz} > k_{xy}$ and by analogous arguments as before, yz is connected by a walk in $\mathbb{S}(G)$. Combining the xz -walk and the yz -walk yields a xy -walk in $\mathbb{S}(G)$, a contradiction. Similarly, one treats the case when $N^-[y] = N^-[z]$. Analogously, one shows that Condition (D4) leads to a contradiction.

To summarize, for each dispensable edge xy there is a walk connecting x and y that consists of non-dispensable edges only, and thus $\mathbb{S}(G)$ is connected. \square

Since $\mathbb{S}(G)$ is uniquely defined and in particular entirely in terms of the adjacency structure of G , we have the following immediate consequence of the definition.

Proposition 3.10. *Any isomorphism $\varphi : G \rightarrow H$, as a map $V(G) \rightarrow V(H)$, is also an isomorphism $\varphi : \mathbb{S}(G) \rightarrow \mathbb{S}(H)$*

4. Algorithms

By Theorem 2.2, every finite simple connected digraph has a unique representation as a strong product of prime digraphs, up to isomorphism and the order of the factors. We shortly summarize the top-level control structure of the

Algorithm 1 Cartesian Skeleton

- 1: **INPUT:** A connected thin digraph $G = (V, E)$;
 - 2: **for** each edge $xy \in E$ **do**
 - 3: Check the dispensability conditions (D1) – (D5). Compute the set D of dispensable edges in H ;
 - 4: **end for**
 - 5: $\mathbb{S}(G) \leftarrow (V, E \setminus D)$
 - 6: **OUTPUT:** The Cartesian skeleton $\mathbb{S}(G)$;
-

Algorithm 2 PFD of *thin* digraphs w.r.t. \boxtimes

- 1: **INPUT:** a connected S -thin digraph G
 - 2: Compute the Cartesian skeleton $\mathbb{S}(G)$ with Algorithm 1
 - 3: Compute the Cartesian PFD of $\mathbb{S}(G) = \boxtimes_{i \in I} H_i$ with the algorithm of Feigenbaum [3]
 - 4: Find all minimal subsets J of I such that the H_J -layers of $H_J \boxtimes H_{I \setminus J}$ where $H_J = \boxtimes_{j \in J} H_j$ and $H_{I \setminus J} = \boxtimes_{j \in I \setminus J} H_j$ correspond to layers of a factor of G w.r.t. the strong product
 - 5: **OUTPUT:** The prime factors of G ;
-

Algorithm 3 PFD of digraphs w.r.t. \boxtimes

- 1: **INPUT:** a connected digraph G
 - 2: Compute $G = G' \boxtimes K_l$, where G' has no nontrivial factor isomorphic to a complete graph K_r ;
 - 3: Determine the prime factorization of K_l , that is, of l ;
 - 4: Compute $H = G'/S$;
 - 5: Compute PFD and prime factors H_1, \dots, H_n of H with Algorithm 2
 - 6: By repeated application of Lemma 4.4, find all minimal subsets J of $I = \{1, 2, \dots, n\}$ such that there are graphs A and B with $G = A \boxtimes B$, $A/S = \boxtimes_{i \in J} H_i$ and $B = \boxtimes_{j \in I \setminus J} H_j$. Save A as prime factor.
 - 7: **OUTPUT:** The prime factors of G ;
-

algorithm for the computation of the PFD. We first compute for a given digraph G the Relation S and its quotient graph G/S . By Lemma 2.4 the digraph G/S is thin and thus, the Cartesian skeleton $\mathbb{S}(G/S)$ is uniquely determined. The key idea is then to find the PFD of G/S w.r.t. the strong product, which is achieved by computing the PFD its Cartesian skeleton $\mathbb{S}(G/S)$ w.r.t. the Cartesian product and to construct the prime factors of G/S using the information of the PFD of $\mathbb{S}(G/S)$. Finally, the prime factors of G/S need to be checked and in some cases be combined and modified, in order to determine the prime factors of the digraph G w.r.t. strong product, see Figure 2 and 3 for examples.

We explain in the following the details of this approach more precise. We start with the construction of the Cartesian skeleton (Algorithm 1) and the computation of the PFD w.r.t. the Cartesian product of digraphs in Section 4.1. We continue to give algorithms for determining the prime factors of thin digraphs (Algorithm 2) in Section 4.2 and non-thin digraphs (Algorithm 3) in Section 4.2. Note, these algorithms are only simple generalizations of the Algorithms 24.3, 24.6 and 24.7 for undirected graphs given in [7]. The novel improvement for directed graphs is the unique construction of the Cartesian skeleton. Therefore, we will refer in most parts of the upcoming proofs to results established in [7] rather than to replicate the proofs.

4.1. Algorithmic Construction of $\mathbb{S}(G)$ and the PFD of Digraphs w.r.t. \boxtimes

Proposition 4.1. *For a thin connected digraph $G = (V, E)$ with maximum degree Δ , Algorithm 1 computes the Cartesian skeleton $\mathbb{S}(G)$ in $O(|E|\Delta^3)$ time.*

Proof. By the arguments given in Section 3 the Algorithm is correct.

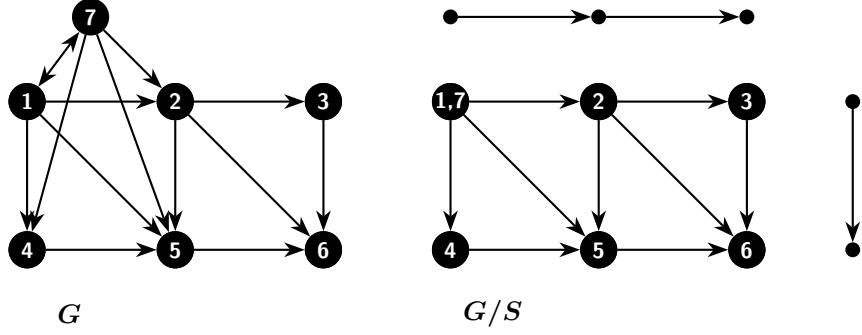


Figure 2: The digraph G is prime. However, the quotient graph G/S has a non-trivial product structure. Hence, the prime factors of G/S must be combined, in order to find the prime factors of G .

To determine the time complexity, note that for any edge $xy \in E$ one of the Conditions (D1) – (D5) is satisfied for some $z_1, z_2 \in V$ if $z_1, z_2 \in N^-[x] \cup N^+[x]$ (and also $z_1, z_2 \in N^-[y] \cup N^+[y]$), see Remark 1. This implies that there are at most $O(|E|\Delta)$ dispensability checks to do. Moreover, several intersection and subset relations of neighborhoods have to be computed, which can all be done in $O(\Delta^2)$ since each neighborhood contains at most Δ elements. Hence, the overall time complexity of the for-loop is $O(|E|\Delta^3)$, while Line 5 can be executed in constant time. \square

The PFD of a connected digraph $G = (V, E)$ w.r.t. the Cartesian product is unique and can be computed in $O(|V|^2 \log_2(|V|)^2)$ time, see the work of Feigenbaum [3]. The algorithm of Feigenbaum works as follows. First one computes the PFD w.r.t. Cartesian product of the underlying undirected graph. This can be done with the Algorithm of Imrich and Peterin in $O(|E|)$ time, [19]. It is then checked whether there is a conflict in the directions of the edges between adjacent copies of the factors, which also determines the overall time complexity. If there is some conflict, then different factors, need to be combined. The latter step is repeated until no conflict exists.

Proposition 4.2 ([3]). *For a connected digraph $G = (V, E)$ the algorithm of Feigenbaum computes the PFD of G w.r.t. the Cartesian product in $O(|V|^2(\log_2 |V|)^2)$ time.*

4.2. Factoring thin Digraphs w.r.t. \boxtimes

We are now interested in an algorithmic approach for determining the PFD of connected thin digraphs w.r.t. strong product, which works as follows. For a given thin connected digraph G one first computes the unique Cartesian skeleton $\mathbb{S}(G)$. This Cartesian skeleton is afterwards factorized with the algorithm of Feigenbaum [3] and one obtains the Cartesian prime factors of $\mathbb{S}(G)$. Note, for an arbitrary factorization $G = G_1 \boxtimes G_2$ of a thin digraph G , Proposition 3.6 asserts that $\mathbb{S}(G_1 \boxtimes G_2) = \mathbb{S}(G_1) \square \mathbb{S}(G_2)$. Since $\mathbb{S}(G_i)$ is a spanning graph of G_i , $i = 1, 2$, it follows that the $\mathbb{S}(G_i)$ -layers of $\mathbb{S}(G_1) \square \mathbb{S}(G_2)$ have the same vertex sets as the G_i -layers of $G_1 \boxtimes G_2$. Moreover, if $\boxtimes_{i \in I} G_i$ is the unique PFD of G then we have $\mathbb{S}(G) = \square_{i \in I} \mathbb{S}(G_i)$. Since $\mathbb{S}(G_i)$, $i \in I$ need not to be prime with respect to the Cartesian product, we can infer that the number of Cartesian prime factors of $\mathbb{S}(G)$, can be larger than the number of the strong prime factors. Hence, given the PFD of $\mathbb{S}(G)$ it might be necessary to combine several Cartesian factors to get the strong prime factors of G . These steps for computing the PFD w.r.t. the strong product of a thin digraph are summarized in Algorithm 2.

Proposition 4.3. *For a thin connected digraph $G = (V, E)$ with maximum degree Δ , Algorithm 2 computes the PFD of G in $O(|V|^2(\log_2 |V|)^2 \Delta + |E|\Delta^3)$ time.*

Proof. Note, Algorithm 2 is a one-to-one analog of the algorithm for the PFD of undirected thin graphs, see [7, Alg. 24.6]. The proof of correctness in [7, Thm 24.9] for undirected graphs depends on the analogue of Lemma 2.3 and the unique construction of the Cartesian skeleton $\mathbb{S}(G)$ for the undirected case. Thus, using analogous arguments for directed graphs as in [7, Section 24.3] we can conclude that Algorithm 2 is correct.

For the time complexity, observe that the Cartesian skeleton $\mathbb{S}(G)$ can be computed in $O(|E|\Delta^3)$ time and the PFD of $\mathbb{S}(G)$ in $O(|V|^2(\log_2 |V|)^2)$ time. We are left with Line 4 and refer to [7, Section 24.3], where the time complexity of

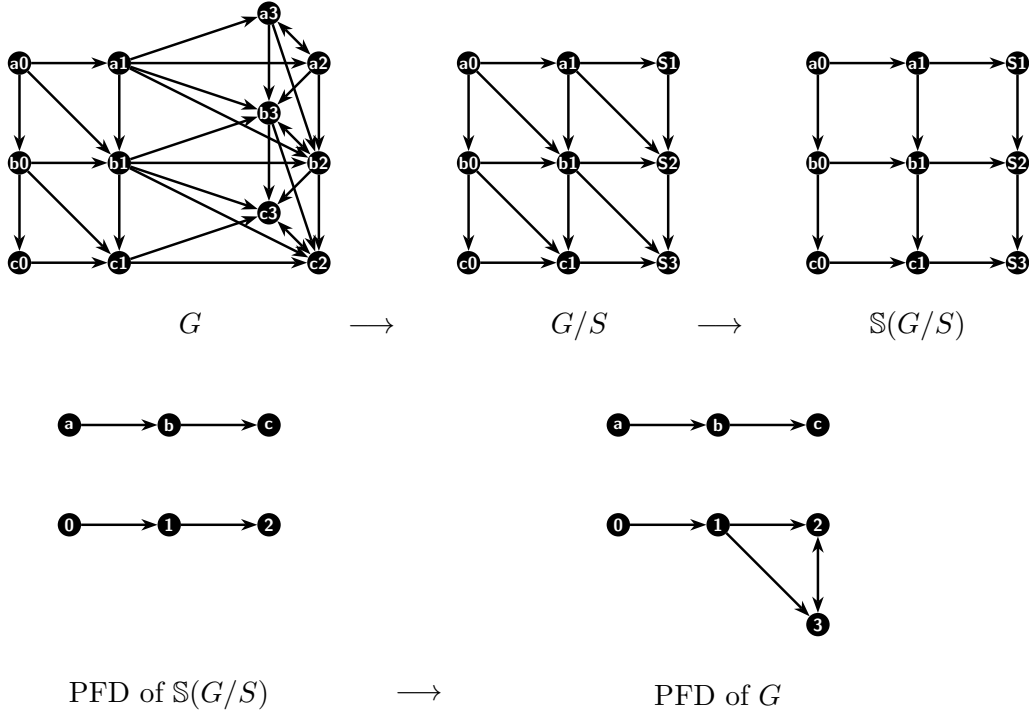


Figure 3: Illustrated are the basic steps of the PFD of strong product of digraphs.

this step is determined with $O(|E||V|\log_2|V|)$. Since $|E| \leq |V|\Delta$, we can conclude that $|E||V|\log_2|V| \leq |V|^2\log_2|V|\Delta$. Thus, we end in overall time complexity of $O(|V|^2(\log_2|V|)^2\Delta + |E|\Delta^3)$. \square

4.3. Factoring non-thin Digraphs w.r.t. \boxtimes

We are now interested in an algorithmic approach for determining the PFD of connected non-thin digraphs w.r.t. strong product, which works as follows. Given an arbitrary digraph G , one first extracts a possible complete factor K_l of maximal size, resulting in a graph G' , i.e., $G \cong G' \boxtimes K_l$, and computes the quotient graph $H = G'/S$. This graph H is thin and the PFD of H w.r.t. the strong product can be computed with Algorithm 2. Finally, given the prime factors of H it might be the case that factors need to be combined to determine the prime factors of G' , see Figure 2. This can be achieved by repeated application of Lemma 4.4. Since $G \cong G' \boxtimes K_l$, we can conclude that the prime factors of G are then the prime factors of G' together with the complete factors K_{p_1}, \dots, K_{p_j} , where $p_1 \dots p_j$ are the prime factors of the integer l . This approach is summarized in Algorithm 3. For an illustrative example see Figure 3.

Lemma 4.4. *Suppose that it is known that a given digraph G that does not admit any complete graphs as a factor is a strong product graph $G_1 \boxtimes G_2$, and suppose that the decomposition $G/S = G_1/S \boxtimes G_2/S$ is known. Then G_1 and G_2 can be determined from G , G_1/S and G_2/S .*

In fact, if $D(x_1, x_2)$ denotes the size of the S -equivalence class of G that is mapped into $(x_1, x_2) \in G_1/S \boxtimes G_2/S$, then the size $D(x_1)$ of the equivalence class of G_1 mapped into $x_1 \in G_1/S$ is $\gcd\{D(x_1, y) \mid y \in V(G_2)\}$. Analogously for $D(x_2)$.

Proof. Invoking Lemma 2.3, 2.4, 2.5, and 2.6, the assertion can be implied by the same arguments as in the proof for undirected graphs [16, Lemma 5.40]. \square

Proposition 4.5. *For a connected digraph $G = (V, E)$ with maximum degree Δ , Algorithm 3 computes the PFD of G in $O(|V|^2(\log_2|V|)^2\Delta + |E|\Delta^3)$ time.*

Proof. Again note, Algorithm 3 is a one-to-one analog of the algorithm for the PFD of undirected thin graphs, see [7, Alg. 24.7]. The proof of correctness in [7, Thm 24.12] for undirected graphs depends on the analogue of Lemma 2.3, 2.4, 2.5, 2.6 and the correctness of Algorithm 2 for the undirected case. Thus, using analogous arguments for directed graphs as in [7, Section 24.3 and Thm. 24.12], we can conclude the correctness of Algorithm 3.

For the time complexity, to extract complete factors K_I , its PFD and the computation of the quotient graph G/S we refer to [7, Lemma 24.10] and conclude that Line 2-3 run in $O(|E|)$ time. The PFD of G/S w.r.t. the strong product can be computed in $O(|V|^2(\log_2 |V|)^2\Delta + |E|\Delta^3)$ time. We are left with Line 6 and again refer to [7, Section 24.3], where the time complexity of this step is determined with $O(|E||V|\log_2 |V|)$. \square

5. Summary and Outlook

We presented in this paper the first polynomial-time algorithm that computes the prime factors of digraphs. The key idea for this algorithm was the construction of a unique Cartesian skeleton for digraphs. The PFD of the Cartesian skeleton w.r.t. the Cartesian product was utilized to find the PFD w.r.t. the strong product of the digraph under investigation.

Since the strong product of digraphs is a special case of the so-called direct product of digraphs, we assume that this approach can also be used to extend the known algorithms for the PFD w.r.t. the direct product [17, 18]. The main challenge in this context is a feasible construction of a so-called Boolean square, in which the Cartesian skeleton is finally computed [6].

Moreover, we strongly assume that the definition of the Cartesian skeleton can be generalized in a natural way for the computation of the strong product of di-hypergraphs in a similar way as for undirected hypergraphs in [13, 14].

Finally, since many graphs are prime although they can have a product-like structure, also known as *approximate* graph products, the aim is to design algorithms that can handle such “noisy” graphs. Most of the practically viable approaches are based on *local* factorization algorithms, that cover a graph by factorizable small patches and attempt to stepwisely extend regions with product structures [9, 10, 8, 12, 11]. Since the construction of the Cartesian skeleton works on a rather local level, i.e., the usage of neighborhoods, we suppose that our approach can in addition be used to establish local methods for finding approximate strong products of digraphs.

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